Solutions seminar exercise 6 (exam questions ECON 4930 Autumn 2007)

1a) (Lecture slides 1, 2,slide 3, 4, 5, Lecture 13, slides 5,6)

No discounting: horizon is so short that discounting makes very little difference. But easy to add.

The Lagrangian

$$L = \sum_{t=1}^{T} \int_{z=0}^{e_t^H} p_t(z)\,dz$$
$$-\sum_{t=1}^{T} \lambda_t (R_t - R_{t-1} - w_t + e_t^H)$$
$$-\sum_{t=1}^{T} \gamma_t (R_t - \bar{R})$$

The Kuhn – Tucker first-order conditions

$$\frac{\partial L}{\partial e_t^H} = p_t(e_t^H) - \lambda_t \le 0 \;(= 0 \text{ for } e_t^H > 0)$$
$$\frac{\partial L}{\partial R_t} = -\lambda_t + \lambda_{t+1} - \gamma_t \le 0 \;(= 0 \text{ for } R_t > 0)$$
$$\lambda_t \ge 0 (= 0 \text{ for } R_t < R_{t-1} + w_t - e_t^H)$$
$$\gamma_t \ge 0 (= 0 \text{ for } R_t < \bar{R}) \;, \qquad t = 1,..,T$$

Shadow price interpretations: Change in the objective function when the constraint changes marginally; changes in exogenous variables w_t (in general relaxation of the constraint) and $\bar{R}$.

Assumptions: $e_t^H > 0$, $p_t(e_t^H) > 0$ (non-satiation), implying

$$p_t(e_t^H) = \lambda_t > 0, t = 1,..,T$$

If no binding reservoir constraint: $\gamma_t = 0$, if a binding reservoir constraint $\gamma_t \ge 0$.

Implication not binding: $p_t(e_t^H) = \lambda_t$, binding $\lambda_t = \lambda_{t+1} - \gamma_t > 0$.

1b) (Lecture slides 2)

Kuhn –Tucker:

$$p_T(e_T^H) = \lambda_T \; (e_T^H > 0),$$
$$-\lambda_T - \gamma_T \le 0 \; (R_T = 0)$$

Non-satiation of demand: $p_T(e_T^H) > 0$

Qualitative conclusions:

$$R_T = 0, \gamma_T = 0, \; p_T(e_T^H) = \lambda_T > 0$$

Zero level in the reservoir at the end of period T follows from the terminal condition and positive price.

1c) (Lecture slides 3, slide 8)

The Kuhn – Tucker conditions fulfilling the conditions given in the question:

$$p_t(e_t^H) = \lambda_t,\ p_{t+1}(e_{t+1}^H) = \lambda_{t+1}$$
$$-\lambda_t + \lambda_{t+1} - \gamma_t = 0\ (R_t > 0)$$
$$\gamma_t \geq 0 (R_t = \bar{R}) \Rightarrow \lambda_t \leq \lambda_{t+1}, p_t \geq p_{t+1}$$

Shadow price on binding reservoir constraint is typically positive implying $\lambda_t < \lambda_{t+1}$.

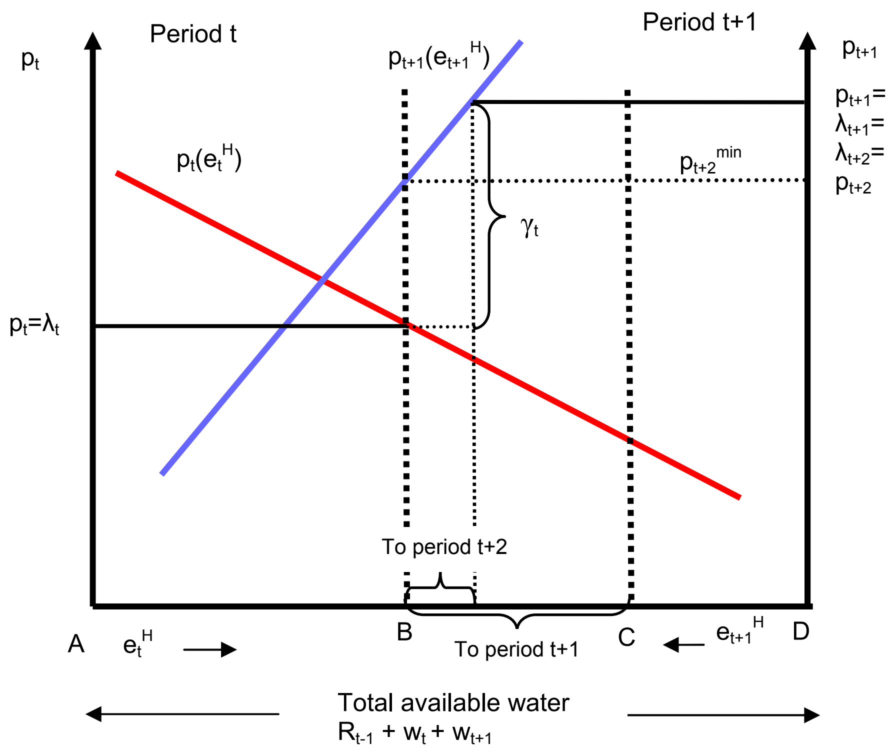
Transfer of water from t+1 to t+2:

$e_{t+1}^H < \bar{R} + w_{t+1}$ and $p_{t+1}(e_{t+1}^H) = p_{t+2}$

The price from the future must be high enough to generate a positive transfer from period t+1 to period t+2. *The limiting value giving zero transfer*:

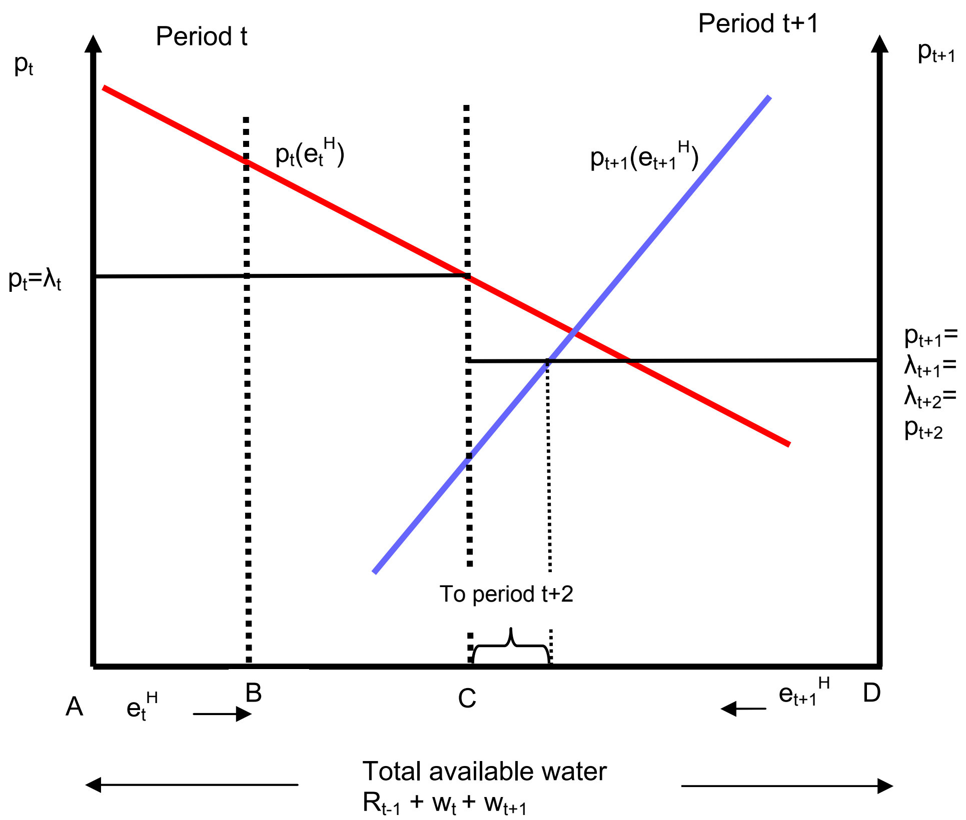
$$p_{t+2}^{\min} = p_{t+1}(\bar{R} + w_{t+1})$$

Illustration: Notation for available water period t is AC. Reservoir size is BC and inflow in period t+1 is CD, like in the book and all lecture slides. Price may be constant from T to t+1, must then have reservoir between empty and full for all periods from T back to t+1. At least the price in period t+2 is given from the future (we must have this price equal to the water



value in period t+2 if the price is constant back from T). To get transfer of water from t+1 to t+2 then the price from the future must be greater than the price p_{t+2}^{min} that would empty the reservoir in period t+1. The correspondence between the Kuhn – Tucker conditions above and the figure should be made explicit by checking relevant intersections and the determination of prices and shadow prices. It may be pointed out that the shadow price on the reservoir constraint is positive if there is a price change.

1d) (See Chapter 3, discussion of figure 3.5, $p_T < p_T^{min}$, Lecture 13, slide 8)
Empty in period t implies that the price from the future must be lower than the price that empties the reservoir. Note that the price from the future must be such that some water is transferred to period t+2. The trick is to see, using the complementary slackness conditions, that for $R_t = 0$ we have $\gamma_t = 0$ and $-\lambda_t + \lambda_{t+1} \le 0$. With $\lambda_{t+1} < p_t = \lambda_t$ we must have $\lambda_t > \lambda_{t+1}$.
Kuhn – Tucker conditions and the illustration: same as under c), check intersections and prices.



1e) (Lecture 13)
Production constraint: (slides 9,10,11) increase in price in period 2 if the constraint becomes binding with no spill, less water available than what is wanted in period 2 by the

unconstrained optimal solution, price must increase in period 2. More water consumed in period 1 because less can be transferred to period 2 due to the production constraint.

Transmission: (slides 23,24) price highest in the high-demand period due to higher marginal loss on lines, must assume that all consumer nodes have the same period as high/low demand. The candidate may add that congestion of lines may happen in high-demand periods, adding to the price difference.

Uncertainty: (Slides 25,26,27,28) Price in period 1 equal to expected price in period 2 if interior solution of the problem max area under demand curve for period 1 and expected area for period 2, but when period 2 comes and inflow is realised actual price may then differ from expected price. Same mechanism with corner solutions, I do not expect corner solutions to be discussed. If all water transferred then expected price will be greater that price in period 1, otherwise max transfer will not be optimal, and vice versa for the other corner solution of zero transfer; lower price is expected than price in period 1 following using all water in period 1. But in period 2 the actual price will again typically differ from expected price.

2a) (Lecture slides 9, Chapter 8, Lecture slides 13, slides 29, 30, 31)
Difference objective functions: (Lecture 9, slide 3) Social planner maximises consumer plus producer surplus = area under the demand curves. Monopolist maximises revenue, i.e., area under the price lines; producer surplus.

Kuhn – Tucker: Lecture 9, slides 8, 9, 10)
The Lagrangian:

$$L = \sum_{t=1}^{T} p_t(e_t^H) \cdot e_t^H$$
$$-\sum_{t=1}^{T} \lambda_t (R_t - R_{t-1} - w_t + e_t^H)$$
$$-\sum_{t=1}^{T} \gamma_t (R_t - \bar{R})$$

Kuhn – Tucker conditions:

$$\frac{\partial L}{\partial e_t^H} = p_t'(e_t^H)e_t^H + p_t(e_t^H) - \lambda_t \le 0 \ \ (= 0 \text{ for } e_t^H > 0)$$
$$\frac{\partial L}{\partial R_t} = -\lambda_t + \lambda_{t+1} - \gamma_t \le 0 \ (= 0 \text{ for } R_t > 0)$$
$$\lambda_t \ge 0 (= 0 \text{ for } R_t < R_{t-1} + w_t - e_t^H)$$
$$\gamma_t \ge 0 (= 0 \text{ for } R_t < \bar{R}) , \qquad t = 1,..,T$$

Reasonable assumptions: $e_t^H > 0, p_t(e_t^H) > 0$.

Price flexibility:

$$\breve{\eta}_t = El_{e_t^H}\, p_t(e_t^H) = \frac{\partial p_t(e_t^H)}{\partial e_t^H}\frac{e_t^H}{p_t}$$

Optimality condition:

$$p_t(e_t^H)(1+\breve{\eta}_t) = \lambda_t$$

Social solution: $p_t(e_t^H) = \lambda_t$ NB! Different lambda for monopoly and social solution

Range for price flexibility: $\breve{\eta}_t \in [-1, 0)$; must have non-negative marginal revenue and derivative of demand function negative.

2b) (lecture 9, slides 10, 11, 12)

The monopolist will vary the production over the periods such that different price levels are realised in accordance with differences of price flexibilities.

Price expression:

$$p_t(e_t^H)(1+\breve{\eta}_t) = \lambda_t = \lambda \Rightarrow p_t = \frac{1}{1+\breve{\eta}_t}\lambda$$

Because all reservoir shadow prices are zero the water value becomes equal for all periods. The price varies inversely with the price flexibility over its range, the more inelastic demand (steep demand curve, absolute value of $\breve{\eta}_t$ high) the higher the price. The water is shifted from periods with relatively inelastic demand to periods with relatively elastic demand. NB! The elasticity reference is to the optimal quantity points on the demand functions, this is a local property of the demand function.

2c) (lecture 9, slide 13, Lecture 10, slide 2,3,4)

Spill in period 1:

$p_1(e_1^H)(1+\breve{\eta}_1)=\lambda_1=0$ (complementary slackness condition used)

No spill in period 2:

$p_2(e_2^H)(1+\breve{\eta}_2)=\lambda_2>0 \Rightarrow \gamma_1>0$

Positive water value in period 2 is the typical case (we may have an intersection in point B for period 2 making the water value and shadow price on the reservoir constraint zero.) Since there is no spill in period 2 a maximal transfer from period 1 to period 2 is optimal, i.e., the reservoir must be constrained in period 1 and the shadow price on the reservoir constraint must typically be positive if the water value in period 1 is zero and positive in period 2.

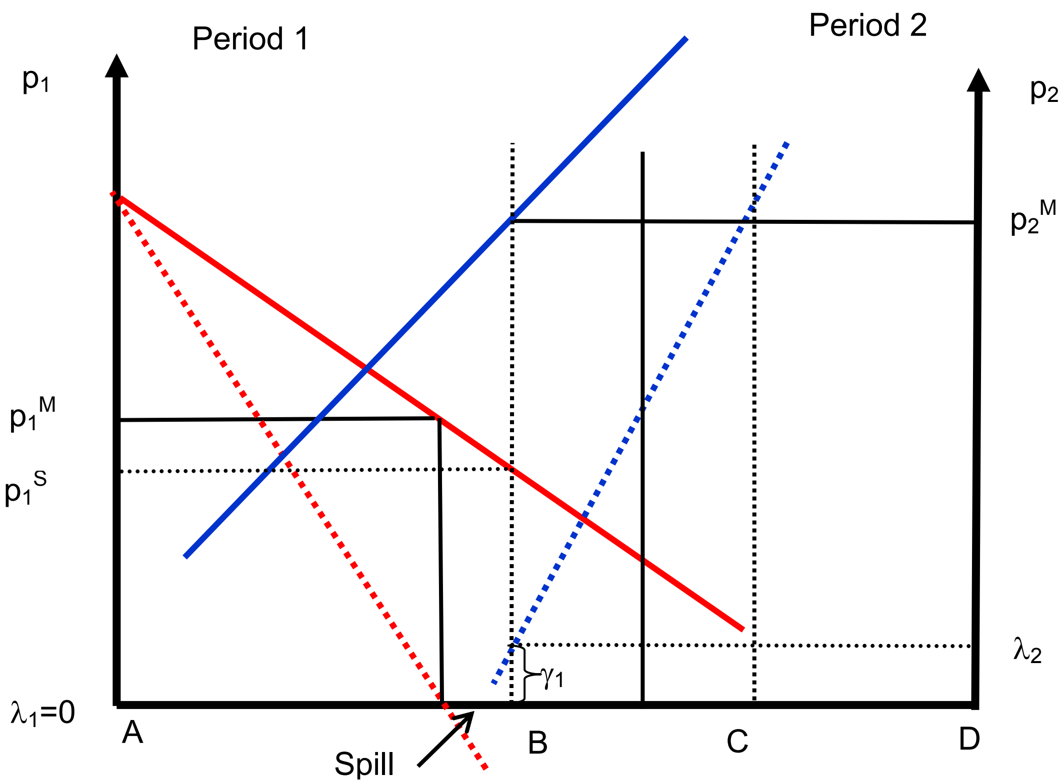
The demand flexibility in period 1: $p_1(e_1^H)(1+\breve{\eta}_1)=0 \Rightarrow \breve{\eta}_1=-1$

This is the steepest the demand function is allowed to be for a unique solution.

Period 2 relatively more elastic: $p_2(e_2^H)(1+\breve{\eta}_2)=\lambda_2>0 \Rightarrow \breve{\eta}_2>-1$

Water is now reduced in the period with the steepest demand curve (locally), apparently in conflict with shifting water to a relatively elastic period from a relatively inelastic period: This rule assumes no spill and binding reservoir constraints. We have spill and a binding constraint and then get other results.

Bathtub illustration, two periods, spill and a binding reservoir constraint in period 1:



By assumption we have a binding reservoir constraint also for the social case. Spill makes the monopoly price in period 1 higher than the social price. Price in period 2 is the same as for the social solution. The local property of elastic-inelastic demand is of no consequence for the results. Spilling in period 1 and max transfer to period 2 determines the prices via the demand functions independently of local flexibility properties.
Correspondence with the Kuhn – Tucker conditions: pointing out that the equations above are obeyed by the bathtub illustration.

2d) (lecture 10, slides 2, 3, 4)
Prohibiting spilling: The water accumulation constraint now becomes an equality in the optimisation problem. We will get a solution for the monopolist that is identical with the social solution except for the values of the shadow prices on stored water and the reservoir constraint. The monopolist's water value in period 1 will become negative. The shadow price on the reservoir constraint will increase and becomes the absolute sum of the water values. Comparing with the social solution: We get the same prices and quantities, only differences are water values and shadow price on the reservoir constraint.